Bayesian Statistical Framework to Construct Probabilistic Models for the Elastic Modulus of Concrete

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Abstract: The commonly used Pauw's formula to predict elastic modulus of concrete is very general and does not address the complexity of modern concretes, such as high-strength concrete, use of different types of aggregates and admixtures, etc. This paper develops a statistical framework to construct probabilistic models for the elastic modulus of concrete and evaluates the influence of different aggregate types, based on a large number of experimental data. The proposed framework to construct probabilistic models expands upon Pauw's formula and properly accounts for both aleatory and epistemic uncertainties. Bayesian updating is used to assess the unknown model parameters based on experimental data. A Bayesian stepwise deletion process is used to identify important explanatory functions and construct parsimonious models. As an application, the approach is used to develop a probabilistic model for concretes made using crushed limestone and crushed quartz schist coarse aggregates.

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Introduction

In single-phase solids, i.e., homogeneous materials, a direct relationship exists between density and modulus of elasticity. In heterogeneous, multiphase materials, i.e., concrete, the volume fraction, density, and modulus of elasticity of each phase, and the characteristics of interfacial transition zone (ITZ) determine the elastic behavior of the composite. Measuring the elastic modulus of concrete, E_c, requires cylinder sample preparation and conducting uniaxial testing on those cylinders in the laboratory. The elastic modulus is given by the shape of the stress–strain curve for concrete under uniaxial loading. Since the curve for concrete is nonlinear, different methods such as tangent modulus, chord modulus, secant modulus, and dynamic modulus are used for computing elastic modulus of concrete. Short of conducting laboratory testing, Pauw's empirical formula (Pauw 1960) is commonly used to predict the elastic modulus of concrete. It is a density-dependent mathematical relation between E_c and compressive strength, f'_c, of concrete. In U.S. customary units it is expressed as $E_c=33w^{3/2}\sqrt{f'_c}$, where w=unit weight of concrete in pounds per cubic feet and f'_c=compressive strength. Both E_c and f'_c are expressed in pounds per square inch. In SI units, Pauw's formula becomes $E_c=43w^{3/2}\sqrt{f'_c}\times 10^{-6}$ (Neville 1995), where E_c is expressed in GPa, f'_c in MPa, and w in kg per cubic meter.

Both the ACI Building Code 318R-83 (ACI 1992) and the ACI Nuclear Safety Structures Code 349-27 (1992) recommend Pauw's formula. The ACI code also recommends a simplified form of Pauw's formula in the case of normal weight and normal strength concrete where the dependence of E_c on w is not accounted for ($E_c=57{,}000\sqrt{f'_c}$, when both E_c and f'_c are expressed in pounds per square inch, and in SI units $E_c=4.73\sqrt{f'_c}$, when E_c is expressed in GPa and f'_c in MPa).

Both the complete and the simplified forms of Pauw's formulas are deterministic and do not account for the uncertainties in the model. Geyskens et al. (1998) quantified the model uncertainties in the simplified Pauw's formula through a comprehensive Bayesian analysis using data on normal weight and normal strength concrete obtained from literature and from further laboratory testing, and developed a model that can be used for probabilistic prediction of E_c.

However, today's concrete is commonly made using chemical and/or mineral admixtures, with variety of aggregate types and curing conditions, all of which will have a great impact on the elastic modulus. Neither of the two Pauw's formulas or the probabilistic model developed by Geyskens et al. (1998) accounts for these factors. In order to examine the influence of different aggregates, admixtures, practices, such as curing, on elastic modulus of concrete, we collected data from the tests performed on variety of conditions in a number of laboratories and constructed a database containing the results of those tests. A Bayesian framework is developed to construct probabilistic models for E_c that properly account for all the prevailing uncertainties, including model errors arising from an inaccurate model form or missing variables, measurement errors, and statistical uncertainty. With the aim of facilitating their use in practice, the probabilistic models are constructed using a model form similar to Pauw's empirical formula with additional correction terms called *explanatory functions*. Methods for assessing the model parameters using the

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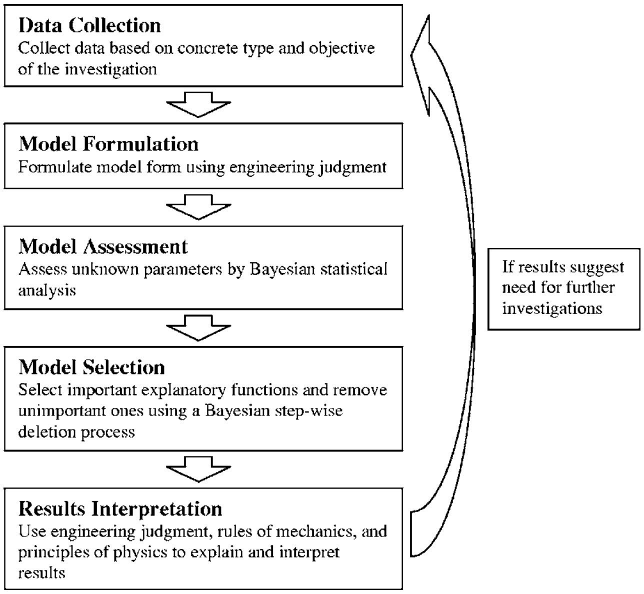

Fig. 1. Steps to construct a probabilistic model

collected data are described. Using a Bayesian step-wise deletion process, explanatory functions that are important to predict E_c are identified and parsimonious probabilistic models are constructed. The identified explanatory functions also provide insight into the underlying behavioral phenomena. Fig. 1 summarizes the key steps in the proposed Bayesian statistical framework.

While the framework described in this paper is aimed at developing probabilistic models for E_c, the approach is general and can be applied to the development and assessment of models in many engineering applications.

As an application, the proposed methodology is used to construct an accurate probabilistic model to predict E_c that can be used in practice for the following two types of concrete:

1. Concrete made using crushed limestone coarse aggregate with volume ranging from 36.6 to 41.3% by volume.
2. Concrete made using crushed quartz schist coarse aggregate with volume ranging from 35.7 to 40% by volume.

The proposed methodology is also used to explore the effect of w on E_c. Tables 1 and 2 list the experimental data for the two types of concrete used in this paper (for the complete database, refer to the database for mechanical properties of concrete at http://bme.t.u-tokyo.ac.jp/researches/detail/concreteDB/index.html).

Probabilistic Models

In the present paper we want to investigate the dependency of E_c on w and f'_c, assessing the most appropriate model form for two types of concrete, and estimating the unknown parameters entering in the selected model.

Following Pauw's empirical formula, the writers consider the general univariate model form

$$\log(E_c) = \theta_1 + \theta_2 \log(f'_c) + \theta_3 \log(w) + \sigma\varepsilon \tag{1}$$

where $\boldsymbol{\Theta}=(\boldsymbol{\theta},\sigma^2)$ denotes the set of unknown model parameters; $\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)$; ε=random variable with zero mean and unit variance; and σ represents the standard deviation of the model error. Note that for given f'_c, w, $\boldsymbol{\theta}$, and σ, we have $\mathrm{Var}[\log(E_c)]=\sigma^2$ as the variance of the model.

Table 1. Experimental Data for Concrete Made Using Crushed Limestone Coarse Aggregate with Volume Ranging from 36.6 to 41.3% by Volume

f'_c (MPa)	E_c (GPa)	w (kg/m^3)
23.73	27.36	2,310.0
25.69	32.56	2,370.0
35.99	33.54	2,320.0
37.17	34.62	2,350.0
45.11	34.72	2,340.0
38.83	35.21	2,383.3
45.70	37.66	2,370.0
43.15	37.72	2,451.0
43.15	37.76	2,450.0
45.80	38.42	2,450.0
45.01	39.47	2,458.0
45.01	39.52	2,460.0
54.92	40.14	2,461.0
57.37	41.78	2,470.0
57.37	41.79	2,474.0
63.94	41.84	2,472.0
72.96	41.87	2,410.0
56.58	41.87	2,433.7
59.33	42.27	2,493.0
59.33	42.27	2,490.0
64.23	42.36	2,460.0
64.23	42.40	2,456.0
66.59	42.53	2,456.0
65.90	43.77	2,453.0
58.64	44.31	2,512.0
89.73	44.33	2,379.9
58.64	44.33	2,510.0
67.86	44.47	2,493.0
72.18	44.52	2,477.0
84.83	44.62	2,450.0
70.31	44.82	2,500.0
70.31	44.85	2,497.0
79.83	45.31	2,490.0
79.83	45.32	2,485.0
105.03	45.60	2,379.9
72.77	45.70	2,510.0
72.77	45.72	2,508.0
69.04	45.88	2,501.0
69.04	45.90	2,500.0
96.99	46.19	2,456.3
66.69	46.28	2,504.0
66.69	46.29	2,500.0
85.91	46.32	2,471.0
79.43	46.39	2,490.0
79.43	46.41	2,487.0

Table 1. (*Continued.*)

f'_c (MPa)	E_c (GPa)	w (kg/m^3)
97.28	46.48	2,427.3
87.18	46.58	2,464.8
91.40	46.68	2,476.9
85.51	46.88	2,481.0
79.83	46.95	2,478.0
99.73	47.37	2,429.2
80.51	47.46	2,510.0
80.51	47.47	2,509.0
116.60	48.54	2,429.2
114.64	48.54	2,379.9
94.83	48.61	2,497.0
97.87	48.64	2,476.9
97.58	48.94	2,500.0
97.58	48.95	2,502.0
100.03	49.33	2,510.0
107.68	49.42	2,489.0
121.80	49.62	2,456.3
122.49	50.11	2,427.3
110.32	50.21	2,476.9
129.35	50.50	2,427.3
127.19	50.90	2,429.2
112.58	50.90	2,500.0
112.58	50.93	2,504.0
100.03	52.00	2,512.0
136.61	52.27	2,456.3

The above additive model form is valid under the following assumptions: (a) the model standard deviation is independent of **x** (homoskedasticity assumption) and (b) the model error has the normal distribution (normality assumption). Employing a suitable transformation of each quantity of interest approximately satisfies these assumptions. For a positive-valued quantity Y, Box and Cox (1964) suggest a parametrized family of variance stabilizing transformations of the form

$$C = \frac{Y^{\lambda} - 1}{\lambda} \quad \lambda \neq 0$$
$$= \ln Y \quad \lambda = 0 \tag{2}$$

where Y denotes the quantity of interest in the original space; and λ=parameter that defines a particular transformation. As special cases, $\lambda=0$ specifies the logarithmic transformation, $\lambda=1/2$ specifies the square-root transformation; $\lambda=1$=linear transformation; and $\lambda=2$ specifies the quadratic transformation. Under the assumptions of homoskedasticity and normality, one can formulate the posterior distribution of λ by using the Bayes' theorem and estimating its value for given data. In many practical situations, the model formulation itself often suggests the most suitable transformation. Diagnostic plots of the data or the residuals against model predictions or individual regressors can be used to verify the suitability of an assumed transformation (Rao and Toutenburg 1997). In the following analyses, considering the non-

Table 2. Experimental Data for Concrete Made Using Crushed Quartz Schist Coarse Aggregate with Volume Ranging from 35.7 to 40% by Volume

f'_c (MPa)	E_c (GPa)	w (kg/m^3)
42.56	24.91	2,367.5
63.84	29.91	2,333.8
66.49	30.01	2,372.1
80.51	30.69	2,367.5
88.85	30.89	2,411.0
80.81	31.87	2,410.9
78.55	31.87	2,418.1
86.00	32.56	2,397.2
86.10	33.44	2,333.8
86.40	35.01	2,372.1
80.51	35.30	2,460.0
97.48	35.60	2,410.9
95.91	36.48	2,333.8
83.65	36.77	2,450.0
109.05	37.17	2,372.1
98.56	37.85	2,460.0
121.99	38.05	2,397.2
100.22	38.15	2,367.5
112.68	38.83	2,411.0
110.32	39.13	2,410.9
107.38	39.23	2,460.0
123.66	39.52	2,411.0
114.25	40.99	2,418.1
128.96	41.97	2,397.2
128.27	43.15	2,418.1

negative nature of E_c, f'_c, and w we used a logarithmic variance stabilizing transformation.

Model Assessment

In assessing a model, or in using a model for prediction purposes, one has to deal with two broad types of uncertainties: aleatory uncertainties (also known as inherent variability or randomness) and epistemic uncertainties (Bedford and Cooke 2001; Gardoni et al. 2002a; Ang and Tang 2006). The former are those inherent in nature; they cannot be influenced by the observer or the manner of the observation. Referring to the model formulations in Eq. (1), this kind of uncertainty is present in the variables E_c, w, f'_c, and partly in the error term ε. The epistemic uncertainties are those that arise from our lack of knowledge, our deliberate choice to simplify the model, from errors that arise in measuring observations, and from the finite size of observation samples. This kind of uncertainty is present in the model parameters $\boldsymbol{\Theta}$ and partly in the error term ε. The fundamental difference between the two types of uncertainties is that the aleatory uncertainties are irreducible while the epistemic uncertainties are reducible, e.g., by use of higher-order models, more accurate measurements and collection of additional data.

Since the model in Eq. (1) is linear in the unknown parameters $\boldsymbol{\theta}$, it can be rewritten as

$$\mathbf{E} = \mathbf{H}\boldsymbol{\theta} + \sigma\boldsymbol{\varepsilon} \tag{3}$$

where $\mathbf{E}$=$n \times 1$ vector of independent observations; n=number of observations; $\mathbf{H}$=$n \times k$ matrix of known regressors or explanatory functions; h_{ij}, $i=1,\ldots,n$ and $j=1,\ldots,k$, $\boldsymbol{\theta}$ denote a $k\times 1$ vector of unknown model parameters; $\boldsymbol{\varepsilon}=n\times 1$ vector of normal random variables having zero mean and unit variance; and σ represents the standard deviation of the model errors. Expanding out the matrices we can write Eq. (3) as

$$\begin{bmatrix}\log(E_c)_1\\ \vdots\\ \log(E_c)_i\\ \vdots\\ \log(E_c)_n\end{bmatrix}=\begin{bmatrix}1 & \log(f'_c)_1 & \log(w)_1\\ \vdots & \vdots & \vdots\\ 1 & \log(f'_c)_i & \log(w)_i\\ \vdots & \vdots & \vdots\\ 1 & \log(f'_c)_n & \log(w)_n\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}+\sigma\begin{bmatrix}\varepsilon_1\\ \vdots\\ \varepsilon_i\\ \vdots\\ \varepsilon_n\end{bmatrix} \quad (4)$$

where, in this case, $k=3$, $h_{i1}=1$, $h_{i2}=\log(f'_c)_i$, and $h_{i3}=\log(w)_i$. As shown by Box and Tiao (1992), the posterior distribution of the unknown model parameters $\boldsymbol{\Theta}$ can be written as

$$p(\boldsymbol{\Theta}|\mathbf{E}) \propto p(s^2|\sigma^2)p(\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\sigma^2)p(\boldsymbol{\Theta}) \quad (5)$$

where

$$\hat{\boldsymbol{\theta}}=(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\mathbf{E}$$

$$s^2=\frac{1}{v}(\mathbf{E}-\hat{\mathbf{E}})'(\mathbf{E}-\hat{\mathbf{E}})$$

$$v=n-k$$

$$\hat{\mathbf{E}}=\mathbf{H}\hat{\boldsymbol{\theta}} \quad (6)$$

Eq. (5) is an expression of the Bayes' theorem, where $p(\boldsymbol{\Theta})$=*prior* distribution that reflects our state of knowledge about $\boldsymbol{\Theta}$ prior to obtaining the experimental observations, $\mathbf{E}$; and $p(\boldsymbol{\Theta}|\mathbf{E})$=*posterior* distribution of $\boldsymbol{\Theta}$ given $\mathbf{E}$. The posterior distribution $p(\boldsymbol{\Theta}|\mathbf{E})$ incorporates both what is known about $\boldsymbol{\Theta}$ before $\mathbf{E}$ are collected and the information content in $\mathbf{E}$. In practice, the prior might incorporate any subjective information about $\boldsymbol{\Theta}$ that is based on our engineering experience and judgment. Following Fisher (1922), $p(s^2|\sigma^2)p(\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\sigma^2)$ can be seen as representing the objective information about $\boldsymbol{\Theta}$ coming from the data and it is called the *likelihood function* of $\boldsymbol{\Theta}$ for given $\mathbf{E}$ and is written as $L(\boldsymbol{\Theta}|\mathbf{E})$.

When new data become available, Eq. (5) can be reapplied to update our present state of knowledge. For example, given an initial sample of observations, $\mathbf{E}_1$, Eq. (5) gives

$$p(\boldsymbol{\Theta}|\mathbf{E}_1) \propto L(\boldsymbol{\Theta}|\mathbf{E}_1)p(\boldsymbol{\Theta}) \quad (7)$$

If a second sample of observations, $\mathbf{E}_2$, distributed independently of $\mathbf{E}_1$, is collected, we can update $p(\boldsymbol{\Theta}|\mathbf{E}_1)$ to account for the new information such that

$$p(\boldsymbol{\Theta}|\mathbf{E}_1,\mathbf{E}_2) \propto L(\boldsymbol{\Theta}|\mathbf{E}_2)L(\boldsymbol{\Theta}|\mathbf{E}_1)p(\boldsymbol{\Theta}) \propto L(\boldsymbol{\Theta}|\mathbf{E}_2)p(\boldsymbol{\Theta}|\mathbf{E}_1) \quad (8)$$

where the posterior distribution in Eq. (7) now plays the role of the prior distribution.

The same updating process can be repeated every time new information becomes available. For example, in case we have m independent samples of observations, the posterior distribution can be written as

$$p(\boldsymbol{\Theta}|\mathbf{E}_1,\ldots,\mathbf{E}_q) \propto p(\boldsymbol{\Theta}|\mathbf{E}_1,\ldots,\mathbf{E}_{q-1})L(\boldsymbol{\Theta}|\mathbf{E}_q) \quad q=2,\ldots,m \quad (9)$$

where $p(\boldsymbol{\Theta}|\mathbf{E}_1)$ is given as in Eq. (7) and is updated after each new sample becomes available. Repeated applications of Bayes' theorem are similar to a learning process, where our present knowledge about the unknown parameters $\boldsymbol{\Theta}$ is updated, as new data become available.

Priors that have no or little information relative to an intended experiment are typically called *noninformative priors*. They reflect the fact that little or nothing is known a priori. Assuming a noninformative prior and with $\boldsymbol{\theta}$ and $\log(\sigma)$ approximately independent and locally uniform, i.e.,

$$p(\boldsymbol{\Theta})=p(\boldsymbol{\theta})p(\sigma^2) \propto \sigma^{-2} \quad (10)$$

we can rewrite the joint posterior distribution in Eq. (5) as (Box and Tiao 1992)

$$p(\boldsymbol{\theta},\sigma^2|\mathbf{E}) \propto p(\sigma^2|s^2)p(\boldsymbol{\theta}|\hat{\boldsymbol{\theta}},\sigma^2) \quad (11)$$

Furthermore, the marginal posterior distribution of σ^2 is in the inverse chi-square distribution, $vs^2\chi_v^{-2}$, and the marginal posterior distribution of $\boldsymbol{\theta}$ is

$$p(\boldsymbol{\theta}|\mathbf{E})=\frac{\Gamma\left(\frac{v+k}{2}\right)|\mathbf{H}'\mathbf{H}|^{1/2}s^{-k}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^k\Gamma\left(\frac{v}{2}\right)(\sqrt{v})^k}\times\left[1+\frac{(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})'\mathbf{H}'\mathbf{H}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})}{vs^2}\right]^{-(v+k)/2}$$

$$-\infty<\theta_i<\infty \qquad i=1,\ldots,3 \quad (12)$$

which is the multivariate t distribution, $t_k[\hat{\boldsymbol{\theta}},s^2(\mathbf{H}'\,\mathbf{H})^{-1},v]$. We note that $\hat{\boldsymbol{\theta}}$=mode and the mean of $\boldsymbol{\theta}$ and its covariance matrix is $vs^2(\mathbf{H}'\mathbf{H})^{-1}/(v-2)$, and the mean and variance of σ^2 are $vs^2/(v-2)$ and $2v^2s^4/[(v-2)^2(v-4)]$, respectively. Derivations and additional detail on these distributions can be found in Box and Tiao (1992).

In the case prior information is available either from engineering judgment or from a previous statistical analysis, computation of the posterior statistics, as well as the normalizing constant κ, is not a simple matter, as it requires multifold integration over the Bayesian kernel, $L(\boldsymbol{\Theta})p(\boldsymbol{\Theta})$. In this case the closed-form solution presented above cannot be used and numerical solutions are the only option. For example, an algorithm for computing the posterior mean and covariance matrix based on importance sampling is described in Gardoni et al. (2002b).

Model Selection

For practical prediction purposes, the selection process should aim at a model that is unbiased, accurate and can be easily adopted in practice. Furthermore, from a statistical standpoint, it is desirable that the model has a parsimonious parametrization (i.e., has as few parameters θ_i as possible) in order to avoid the loss of precision of the estimates and of the model due to the inclusion of unimportant predictors and to avoid overfitting the data.

The model form in Eq. (1) is unbiased by formulation. Furthermore, a good measure of its accuracy is represented by the posterior mean of its standard deviation σ. Specifically, among a set of parsimonious candidate models (in terms of number of unknown parameters θ_i), the one that has the smallest σ can be considered to be the most accurate. Therefore, an estimate of the parameter σ and of its standard deviation, e.g., its posterior mean and posterior standard deviation, can be used to select the most accurate model among several viable candidates.

In Gardoni et al. (2002a), a step-wise deletion procedure is described for reducing the number of parameters in the probabilistic model. The aim is to achieve a compromise between model simplicity (few terms) and model accuracy (small σ). In essence, the procedure eliminates each term when the coefficient of variation (COV) of θ_i is large in comparison to σ. Note that σ is approximately equal to the COV of the predicted E_c for a given set of parameters, therefore, the accuracy of the model is not expected to improve by including a term that has a COV much greater than the model itself. This deletion of terms is carried out step-wise while monitoring the posterior mean of σ to make sure that it does not unduly increase. The step-wise deletion process proceeds as follows:

1. Compute the posterior statistics of the model parameters $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_p)$ and σ.
2. Identify the explanatory function h_j among the higher order terms whose coefficient θ_j has the largest posterior COV. The term h_j=least informative among all the explanatory functions, so one may select to drop it from the model. Deletion is always made from higher order terms to maintain hierarchical submodels. In this application, $h_1=1$ is always present in the model while $h_2=\log(f'_c)$ and $h_3=\log(w)$ may be removed.
3. Assess the reduced model of Step 2 by estimating its parameters. If the posterior mean of σ has not increased by an unacceptable amount, accept the reduced model and return to Step 2 for possible further reduction of the model. Otherwise, the reduction is not desirable and the model form before the reduction is as parsimonious as possible.

While there is considerable room for judgment in the above procedure, this is a part of the art of model building. Applications in the remaining of this paper demonstrate this step-wise model reduction procedure.

Application of the Bayesian Framework

As an application, the methodology described in the previous sections is used to construct two accurate probabilistic models to predict E_c that can be used in practice and to explore the effect of w on E_c. In this application, the following two types of concrete are considered:

1. Concrete made using crushed limestone coarse aggregate with volume ranging from 36.6 to 41.3% by volume.
2. Concrete made using crushed quartz schist coarse aggregate with volume ranging from 35.7 to 40% by volume.

The data used for the model assessment listed in Tables 1 and 2 have been collected from technical papers of the Annual Meetings of the Architectural Institute of Japan (AIJ), Proceedings of the Japan Concrete Institute (JCI), Proceedings of Cement and Concrete from Cement Association of Japan (CAJ), Journal of Cement and Concrete by CAJ, JCI Concrete Journal, etc., published during the last quarter century in Japan.

First, we look at concrete made using crushed limestone coarse aggregate. The experimental data are reported in Table 1. The ITZ, which represents the interfacial region between the particles of coarse aggregate and the hydrated cement paste, is a thin shell around the aggregate and is generally weaker than either of

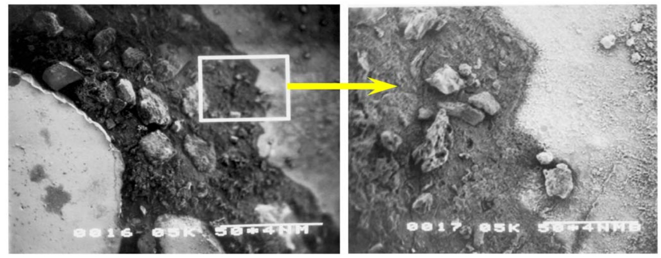

Fig. 2. Influence of mineralogy on microstructure of interfacial transition zone: limestone produces a dense interface zone (Nemati et al. 1997, with permission)

the two components of concrete, and therefore it exercises a far greater influence on the mechanical behavior of concrete than is reflected by its size. The elastic moduli at the ITZ can be as much as 30% lower than in bulk cement paste (Lutz and Monteiro 1995). The quality of ITZ is of importance and may affect the value of E_c when the bond is particularly strong, as is the case in high performance concrete. The mineralogical characteristics of aggregate affect the microstructure of the ITZ. In the case of limestone, there is a chemical reaction between the limestone and the hydrated cement paste and, consequently, a dense interface zone is formed (Nemati and Monteiro 1997), as can be seen in Fig. 2.

Fig. 2 shows scanning electron microscopy (SEM) micrographs of no stress-induced microcracks or regions of connected porosity in the interfacial transition zone, when limestone aggregates are used in concrete. The magnified micrograph on the right shows the ITZ microstructure.

A new probabilistic model that better predicts E_c for this type of concrete is constructed using the model form in Eq. (1). Using a noninformative prior distribution and the experimental data listed in Table 1, the posterior distribution of σ^2 is an inverse chi-square distribution, $\upsilon s^2\chi_{\upsilon}^{-2}$, with $\upsilon=67$ and $s=0.019$. The values of υ and s are computed using Eq. (6). While the posterior statistics of $\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)$ are summarized in Table 3 and compared with the original values according to Pauw's formula.

As described in the Model Selection section, since the largest COV equals 0.067 (for the parameter θ_3) and it is close in magnitude to s all the explanatory functions in the model are important to predict E_c and removing any term would deteriorate the quality of the model.

Comparing the posterior statistics of $\boldsymbol{\theta}$ with the original values in Pauw's empirical formula, we notice that Pauw's formulas overestimates the value of θ_1, θ_2, and θ_3. However, we should note that there are substantial difficulties that arise in interpreting the numerical values of empirical regression coefficients in case of high positive or negative correlation between the parameters.

Table 3. Posterior Statistics of $\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)$ for Concrete Made Using Crushed Limestone Coarse Aggregate with Volume Ranging from 36.6 to 41.3% by Volume

				Correlation coefficient		
Parameter	Original value	Mean	Standard deviation	θ_1	θ_2	θ_3
θ_1	−3.15	−5.29	0.975	1.0	0.39	−0.98
θ_2	0.500	0.265	0.007	0.39	1.0	−0.42
θ_3	1.50	1.90	0.126	−0.98	−0.42	1.0

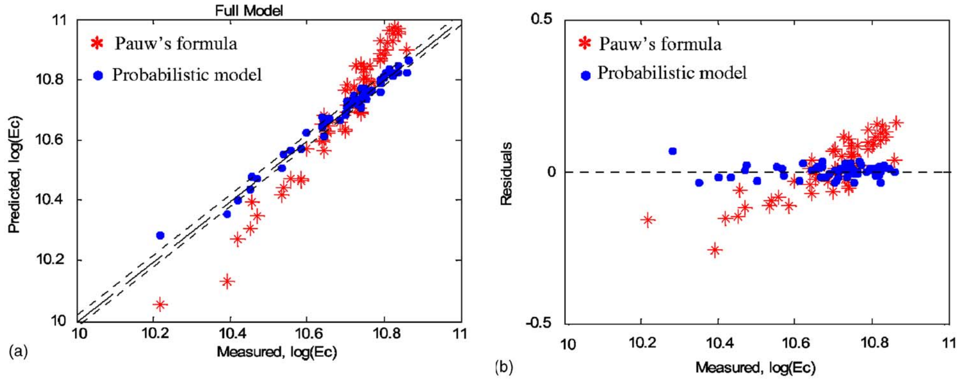

Fig. 3. Comparison between measured and predicted E_c and corresponding residuals based on full (a); reduced (b) models for concrete made using crushed limestone coarse aggregate. Dotted lines delimit the region within one standard deviation of the mean of the probabilistic model.

So, due to the high negative correlation between θ_1 and θ_3, observations on θ_1 and θ_3 may require further experimental investigations.

Fig. 3 shows a comparison between the measured and the predicted values of the elastic module of concrete based on the original Pauw's formula (*) and the mean value of the proposed probabilistic model (•). For a perfect model, the experimental data should line up along the 1:1 dashed line. The dotted lines delimit the region within one standard deviation of the mean of the probabilistic model. We see that Pauw's formula is strongly biased and tends to systematically underestimate E_c for lower values and to overestimate E_c for higher values. A basic tool for detecting deviations from the assumptions made on the probabilistic models and for examining the quality of the fit is diagnostic plots of the residuals. A systematic pattern in the residuals plotted versus the fitted values would suggest deviation from linearity in $\boldsymbol{\theta}$ and, hence, an inadequate fit (Rao and Toutenburg 1997). We see that the residuals plotted versus the measured data for original Pauws' formula (*) show a clear linear pattern. While for the proposed model, the residuals are randomly distributed showing no sign of a pattern and, therefore, of departure from linearity, which supports the quality of the fit.

Fig. 4 plots the predicted E_c using Pauws' formula (*) and using the proposed model (•) versus the measured f_c'. The measured E_c are also shown in Fig. 4 (O). We can see that Pauws' formula systematically underestimates E_c for f_c' below a *threshold strength of E_c estimation* ($f_c' < 75$ MPa) and systematically overestimates E_c above it ($f_c' > 75$ MPa). The proposed formula corrects for this systematic errors and is unbiased over the entire range of f_c'.

Next, we consider concrete made using crushed quartz schist coarse aggregate with volume ranging from 35.7 to 40% by volume. The experimental data are reported in Table 2. In case of quartz aggregate, there is an elastic mismatch between aggregate and the bulk cement paste. Aggregates with low modulus of elasticity (that is, a modulus not very different from the modulus of elasticity of hydrated cement paste) lead to lower bond stresses with the matrix, which is beneficial with respect to high performance concrete. The experimental results indicate that the ITZ between quartz particles and hydrated cement paste is always less dense than bulk paste, regardless of the aggregate size (Ping et al. 1991).

As we did for concrete made using crushed limestone coarse aggregate, we construct a new probabilistic model that better predicts E_c for this type of concrete. The probabilistic model is constructed using the model form in Eq. (1). Using a noninformative prior distribution and the experimental data listed in Table 2, the posterior distribution of σ^2 is an inverse chi-square distribution $\upsilon s^2 \chi_{\upsilon}^{-2}$ with $\upsilon = 22$ and $s = 0.046$. The values of υ and s are computed using Eq. (6). Posterior statistics of $\boldsymbol{\theta}$ are summarized in

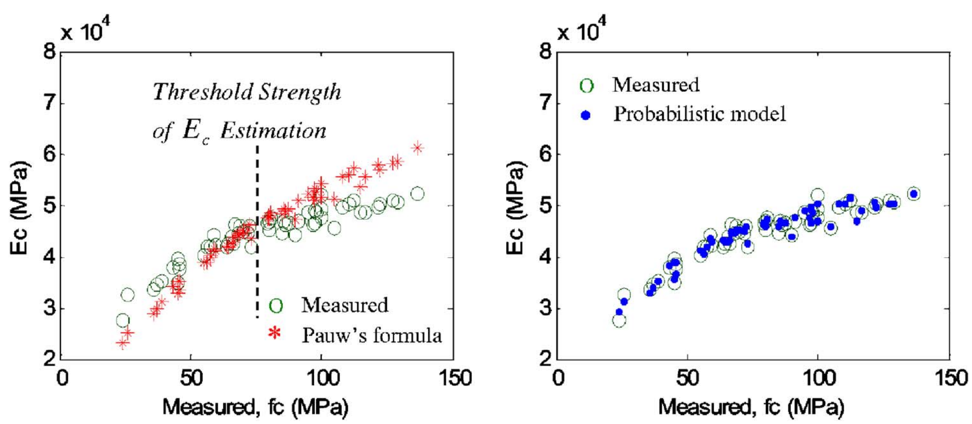

Fig. 4. Predicted E_c using Pauws' formula (*) and using the proposed model (•) versus the measured f_c' for concrete made using crushed limestone coarse aggregate

Table 4. Posterior Statistics of $\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)$ for Concrete Made Using Crushed Quartz Schist Coarse Aggregate with Volume Ranging from 35.7 to 40% by Volume

Parameter	Original values	Mean	Standard deviation	Correlation coefficient θ_1	θ_2	θ_3
θ_1	−3.15	0.830	4.85	1.0	0.28	−0.98
θ_2	0.500	0.462	0.0391	0.28	1.0	−0.31
θ_3	1.50	0.97	0.631	−0.98	−0.31	1.0

Table 5. Posterior Statistics of $\boldsymbol{\theta}=(\theta_1,\theta_2)$ for Concrete Made Using Crushed Quartz Schist Coarse Aggregate with Volume Ranging from 35.7 to 40% by Volume

Parameter	Mean	Standard deviation	Correlation coefficient θ_1	θ_2
θ_1	8.30	0.173	1.0	−0.99
θ_2	0.480	0.038	−0.99	1.0

Table 4 and compared with the original values according to Pauw's formula.

Comparing the posterior statistics of $\boldsymbol{\theta}$ with the original values in Pauw's empirical formula, we notice that Pauw's formulas underestimates the value of θ_1 and overestimates the values of θ_2 and θ_3. Also, the posterior coefficients for concrete made using crushed quartz schist coarse aggregate are different from the coefficient for concrete made using crushed limestone coarse aggregate. As noted earlier, the interpretation of the numerical values of the regression coefficients in case of high positive or negative correlation between the parameters presents some challenges. In particular, due to the high negative correlation between θ_1 and θ_3, observations on θ_1 and θ_3 may require further experimental investigations.

The largest COV equals 0.650 (for the parameter θ_3). Since it is significantly larger than s, following the step-wise deletion process described in the Model Selection section, we drop $\theta_3 \log(w)$ from the model. The reduced model form is then

$$\log(E_c) = \theta_1 + \theta_2 \log(f_c') + \sigma\varepsilon \tag{13}$$

Reassessing the reduced model, we obtain that the posterior distribution of σ^2 is now $\upsilon s^2\chi_\upsilon^{-2}$ with $\upsilon=23$ and $s=0.047$, which indicates no appreciable deterioration of the model accuracy. Table 5 summarize the posterior statistics of $\boldsymbol{\theta}=(\theta_1,\theta_2)$

While in the case of concrete made using limestone aggregate concrete $\log(w)$ is an important explanatory function, in the case of concrete made using quartz schist coarse aggregate, $\log(f_c')$=only explanatory function that is statistically significant. Since the largest COV equals now 0.079 (for the parameter θ_2) is close in magnitude to s, the reduced model is as parsimonious as possible and all the explanatory functions in the model are important to predict E_c. Removing any term would deteriorate the quality of the model.

Similarly to Fig. 3, Fig. 5 shows a comparison between the measured and mean predicted values of E_c. For a perfect model, the experimental data should line up along the 1:1 dashed line. The dotted lines delimit the region within one standard deviation of the mean. A plot of the residuals versus the fitted values shows

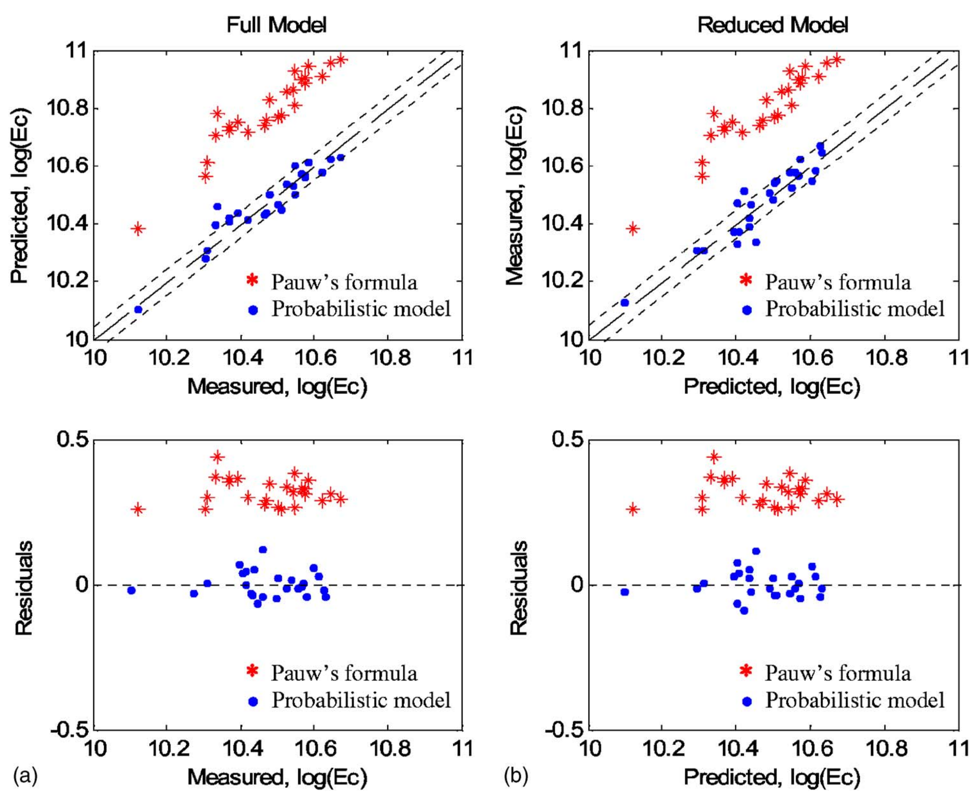

Fig. 5. Comparison between measured and predicted E_c and corresponding residuals based on full (a); reduced (b) models for concrete made using quartz schist coarse aggregate. Dotted lines delimit the region within one standard deviation of the mean of the probabilistic model.

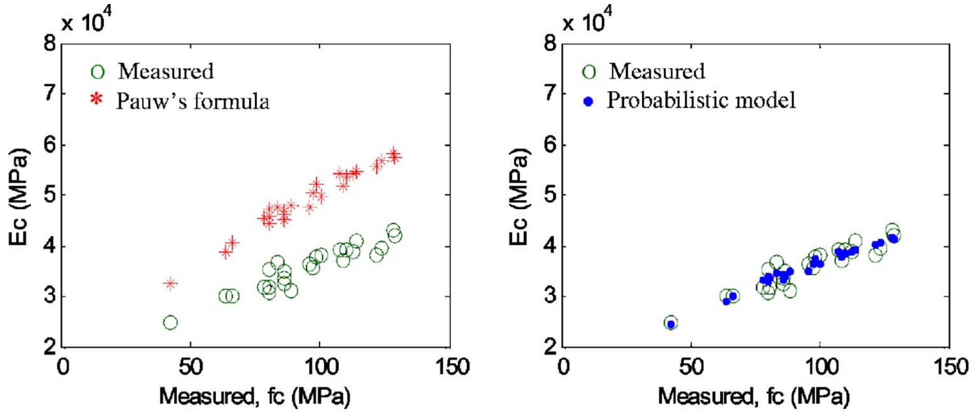

Fig. 6. Predicted E_c using Pauws' formula (*) and using the proposed model (•) versus the measured f_c' for concrete made using quartz schist coarse aggregate

that the residuals are randomly distributed with no sign of a pattern and, therefore, of departure from linearity, supporting the quality of the fit. Also, a comparison of the left and the right plots show that there is no significant deterioration in the model accuracy after removing the explanatory function $\log(w)$.

Similarly to Fig. 4, Fig. 6 plots the predicted E_c using Pauws' formula (*) and using the proposed model (•) versus the measured f_c'. We can see that Pauws' formula in case of concrete made using quartz schist coarse aggregate systematically underestimates E_c over the entire range of f_c'. The proposed formula is unbiased and corrects for the systematic errors in Pauws' formula.

Conclusions

A comprehensive Bayesian framework for constructing probabilistic models for the elastic modulus of concrete is formulated. The models are unbiased and explicitly account for all the prevailing uncertainties. A method for assessing the model parameters using experimental data is described and a Bayesian stepwise deletion process is proposed to select important explanatory functions and construct parsimonious probabilistic models. The identified explanatory functions and the values of their coefficients provide insight into the underlying behavioral phenomena.

Although the Bayesian framework presented in this paper is aimed at developing probabilistic models for the elastic modulus of concrete, the approach is general and can be applied to the development and assessment of models in many engineering applications.

As a practical application, this framework is used to (a) explore the effect of unit weight of concrete, on the elastic modulus of two different concretes (one made using crushed limestone and one made using crushed quartz schist coarse aggregates) and to (b) construct accurate probabilistic models to predict the elastic modulus of concrete that can be used in practice. It is observed that, while for concrete made using crushed limestone coarse aggregate with volume ranging from 36.6 to 41.3% by volume is important to include w in the probabilistic model, for concrete made using crushed quartz schist coarse aggregate with volume ranging from 35.7 to 40% by volume, w can be eliminated from the model form without a significant loss in accuracy of the probabilistic model.

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